

MACAULAY 2 AND THE GEOMETRY OF SCHEMES

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This tutorial illustrates how to use Dan Grayson and Mike Stillman's computer algebra system, MACAULAY2, to study schemes. The examples are taken from the homework for an algebraic geometry class given at the University of California, Berkeley in the fall of 1999. This graduate course was taught by the second author with assistance from the first author. Our choice of problems follows the material in David Eisenbud and Joe Harris' textbook *The Geometry of Schemes*. In fact, four of our ten problems are taken directly from their exercises.

Distinguished open sets. We begin with a simple example involving the definition of an affine scheme; see section I.1.4 in Eisenbud and Harris (1999). This example also indicates some of the subtleties involved in working with arithmetic schemes.

Question 1. *Let $R = \mathbb{Z}[x, y, z]$ and $X = \text{Spec}(R)$; in other words, X is affine 3-space over the integers. Let $f = x$ and consider the basic open subset $X_f \subset X$.*

- (a) *If $e_1 = x+y+z$, $e_2 = xy+xz+yz$ and $e_3 = xyz$ are the elementary symmetric functions then the set $\{X_{e_i}\}_{1 \leq i \leq 3}$ is an open cover of X_f .*
- (b) *If $p_1 = x + y + z$, $p_2 = x^2 + y^2 + z^2$ and $p_3 = x^3 + y^3 + z^3$ are the power sum symmetric functions then $\{X_{p_i}\}_{1 \leq i \leq 3}$ is NOT an open cover of X_f .*

Solution. By Lemma I-6 in Eisenbud and Harris (1999), it suffices to show that e_1, e_2 and e_3 generate the unit ideal in R_f . This is equivalent to showing that x^m belongs to the R -ideal $\langle e_1, e_2, e_3 \rangle$ for some $m \in \mathbb{N}$. In particular, the saturation $(\langle e_1, e_2, e_3 \rangle : x^\infty)$ is the unit ideal if and only if $\{X_{e_i}\}_{1 \leq i \leq 3}$ is an open cover of X_f . MACAULAY2 allows us to work with homogenous ideals over \mathbb{Z} and we obtain:

```
i1 : R = ZZ[x, y, z];
i2 : elementaryBasis = ideal(x+y+z, x*y+x*z+y*z, x*y*z);
o2 : Ideal of R
i3 : saturate(elementaryBasis, x)
o3 = ideal 1
o3 : Ideal of R
```

Similarly, to prove that $\{X_{p_i}\}_{1 \leq i \leq 3}$ is not an open cover of X_f , it is enough to show that $(\langle p_1, p_2, p_3 \rangle : x^\infty)$ is not the unit ideal. We compute this saturation:

```
i4 : powerSumBasis = ideal(x+y+z, x^2+y^2+z^2, x^3+y^3+z^3);
o4 : Ideal of R
i5 : saturate(powerSumBasis, x)
o5 = ideal (6, x + y + z, 2y^2 - y*z + 2z^2, 3y*z)
```

However, working over the field \mathbb{Q} , we find that $(\langle p_1, p_2, p_3 \rangle : x^\infty)$ is the unit ideal.

```
i6 : S = QQ[x, y, z];
i7 : powerSumBasis = ideal(x+y+z, x^2+y^2+z^2, x^3+y^3+z^3);
o7 : Ideal of S
i8 : saturate(powerSumBasis, x)
o8 = ideal 1
o8 : Ideal of S
```

Irreducibility. The study of complex semisimple Lie algebras gives rise to an important family of algebraic varieties called nilpotent orbits. To illustrate one of the properties appearing in section I.2.1 of Eisenbud and Harris (1999), we examine the irreducibility of a particular nilpotent orbit.

Question 2. *Let X be the set of complex 3×3 matrices which are nilpotent. Show that X is an irreducible algebraic variety.*

Solution. A 3×3 matrix M is nilpotent if and only if its minimal polynomial divides T^k , for some $k \in \mathbb{N}$. Since each irreducible factor of the characteristic polynomial of M is also a factor of the minimal polynomial, we conclude that the characteristic polynomial of M is T^3 . It follows that the coefficients of the characteristic polynomial (except for the leading coefficient which is 1) of a generic 3×3 matrix define the algebraic variety X .

To show X is irreducible over \mathbb{C} , it is enough to construct a rational parameterization of X ; see Proposition 4.5.6 in Cox, Little, and O'Shea (1996). To achieve this, observe that $\mathrm{GL}_n(\mathbb{C})$ acts on X by conjugation. Jordan's canonical form theorem implies that there are exactly three orbits; one for each of the following matrices:

$$N_0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, N_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } N_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Each orbit is defined by a rational parameterization, so it suffices to show that the closure of the orbit containing N_2 is the entire variety X . In MACAULAY2, this calculation can be done as follows:

```

i1 : S = QQ[t, y_0..y_8, a..i, MonomialOrder => Eliminate 10];
i2 : N2 = (matrix {{0,1,0},{0,0,1},{0,0,0}}) ** S
o2 = {0} | 0 1 0 |
      {0} | 0 0 1 |
      {0} | 0 0 0 |

          3      3
o2 : Matrix S <--- S
i3 : G = genericMatrix(S, y_0, 3, 3);
          3      3
o3 : Matrix S <--- S

```

To determine the entries in GN_2G^{-1} , we use the classical adjoint to construct the inverse of the generic matrix G .

```

i4 : adj = (G,i,j) -> (
    n := degree target G;
    m := degree source G;
    (-1)^(i+j)*det(submatrix(G, {0..(i-1), (i+1)..(n-1)},
        {0..(j-1), (j+1)..(m-1)}))
);

i5 : classicalAdjoint = (G) -> (
    n := degree target G;
    matrix table(n, n, (i, j) -> adj(G, j, i))
);

i6 : numerators = G*N2*classicalAdjoint(G);
          3      3
o6 : Matrix S <--- S
i7 : D = det(G);
i8 : M = genericMatrix(S, a, 3, 3);
          3      3
o8 : Matrix S <--- S

```

The entries in GN_2G^{-1} give a rational parameterization of the orbit generated by N_2 . We give an “implicit representation” of this variety by using elimination theory; see section 3.3 in Cox, Little, and O’Shea (1996).

```

i9 : elimIdeal = minors(1, (D*id_(S^3))*M-numerators) + ideal(1 - D*t);
o9 : Ideal of S
i10 : closureOfOrbit = ideal(selectInSubring(1, gens gb elimIdeal));
o10 : Ideal of S

```

Finally, we check that the closure of this orbit is equal to X scheme-theoretically.

```

i11 : X = ideal submatrix((coefficients({0},
    det(M - t*id_(S^3))))_1, {1,2,3})
o11 = ideal (a + e + i, b*d - a*e + c*g + f*h - a*i - e*i,
    - c*e*g + b*f*g + c*d*h - a*f*h - b*d*i + a*e*i)
o11 : Ideal of S
i12 : closureOfOrbit == X
o12 = true

```

Singular Points. Section I.2.2 in Eisenbud and Harris (1999) provides the definition of a singular point of a scheme. In our third question, we study the singular locus of a family of elliptic curves. Section V.3 in Eisenbud and Harris (1999) also contains related material.

Question 3. Consider a general form of degree 3 in $\mathbb{Q}[x, y, z]$;

$$F = ax^3 + bx^2y + cx^2z + dxy^2 + exyz + fxz^2 + gy^3 + hy^2z + iyz^2 + jz^3.$$

Give necessary and sufficient conditions in terms of a, \dots, j for the cubic curve $\text{Proj}(\mathbb{Q}[x, y, z]/\langle F \rangle)$ to have a singular point.

Solution. A time consuming elimination gives the degree 12 polynomial which defines the singular locus of a general form of degree 3. This can be done in MACAULAY2 as follows. We have not displayed the output o6, as this discriminant has 2040 terms in the 10 variables a, \dots, j .

```
i1 : S = QQ[x, y, z, a..j, MonomialOrder => Eliminate 2];
i2 : F = a*x^3+b*x^2*y+c*x^2*z+d*x*y^2+e*x*y*z+
      f*x*z^2+g*y^3+h*y^2*z+i*y*z^2+j*z^3;
i3 : partials = submatrix(jacobian matrix{{F}}, {0..2}, {0})
o3 = {1} | 3x2a+2xyb+y2d+2xzc+zye+z2f |
      {1} | x2b+2xyd+3y2g+xze+2yzh+z2i |
      {1} | x2c+xye+y2h+2xzf+2yzi+3z2j |
           3      1
o3 : Matrix S <--- S
i4 : singularities = ideal(partial) + ideal(F);
o4 : Ideal of S
i5 : elimDiscr = ideal selectInSubring(1, gens gb singularities);
o5 : Ideal of S
i6 : elimDiscr = substitute(elimDiscr, {z => 1});
o6 : Ideal of S
```

There is also a simple and more useful determinantal formula for this discriminant. It is a specialization of the formula (2.8) in section 3.2 in Cox, Little, and O'Shea (1998):

```
i7 : hessian = det submatrix(jacobian ideal partials, {0..2}, {0..2});
i8 : A = (coefficients({0,1,2}, submatrix(
      jacobian matrix{{F}}, {0..2}, {0})))_1;
           3      6
o8 : Matrix S <--- S
i9 : B = (coefficients({0,1,2}, submatrix(
      jacobian matrix{{hessian}}, {0..2}, {0})))_1;
           3      6
o9 : Matrix S <--- S
i10 : detDiscr = ideal det (A || B);
o10 : Ideal of S
i11 : detDiscr == elimDiscr
o11 = true
```

Fields of Definition. Schemes over non-algebraically closed fields arise in number theory. Our solution to Exercise II-6 in Eisenbud and Harris (1999) indicates one technique for working over a number field in MACAULAY2.

Question 4. An inclusion of fields $K \hookrightarrow L$ induces a map $\mathbb{A}_L^n \rightarrow \mathbb{A}_K^n$. Find the images in $\mathbb{A}_{\mathbb{Q}}^2$ of the following points of $\mathbb{A}_{\mathbb{Q}}^2$ under this map.

- (a) $\langle x - \sqrt{2}, y - \sqrt{2} \rangle$;
- (b) $\langle x - \sqrt{2}, y - \sqrt{3} \rangle$;
- (c) $\langle x - \zeta, y - \zeta^{-1} \rangle$ where ζ is a 5-th root of unity ;
- (d) $\langle \sqrt{2}x - \sqrt{3}y \rangle$;
- (e) $\langle \sqrt{2}x - \sqrt{3}y - 1 \rangle$.

Solution. The images can be determined by (1) replacing coefficients not belonging to K with indeterminates, (2) adding the minimal polynomials of these coefficients to the given ideal in $\mathbb{A}_{\mathbb{Q}}^2$ and (3) eliminating the new indeterminates. Here are the five examples:

```
i1 : S = QQ[a, b, x, y, MonomialOrder => Eliminate 2];
i2 : Ia = ideal(x-a, y-a, a^2-2);
o2 : Ideal of S
i3 : ideal selectInSubring(1, gens gb Ia)
      2
o3 = ideal (x - y, y  - 2)
o3 : Ideal of S
i4 : Ib = ideal(x-a, y-b, a^2-2, b^2-3);
o4 : Ideal of S
i5 : ideal selectInSubring(1, gens gb Ib)
      2      2
o5 = ideal (y  - 3, x  - 2)
o5 : Ideal of S
i6 : Ic = ideal(x-a, y-a^4, a^4+a^3+a^2+a+1);
o6 : Ideal of S
i7 : ideal selectInSubring(1, gens gb Ic)
      2      2      3      2
o7 = ideal (x*y - 1, x  + y  + x + y + 1, y  + y  + x + y + 1)
o7 : Ideal of S
i8 : Id = ideal(a*x+b*y, a^2-2, b^2-3);
o8 : Ideal of S
i9 : ideal selectInSubring(1, gens gb Id)
      2      3      2
o9 = ideal(x  - -*y )
      2
o9 : Ideal of S
i10 : Ie = ideal(a*x+b*y-1, a^2-2, b^2-3);
```

```

o10 : Ideal of S
i11 : ideal selectInSubring(1, gens gb Ie)
      4      2 2      9 4      2      3 2      1
o11 = ideal(x - 3x y + -*y - x - -*y + -)
              4              2      4
o11 : Ideal of S

```

Multiplicity. The multiplicity of a zero-dimensional scheme X at a point $p \in X$ is defined to be the length of the local ring $\mathcal{O}_{X,p}$. Unfortunately, we cannot work directly in the local ring in MACAULAY2. What we can do, however, is to compute the multiplicity by computing the degree of the component of X supported at p ; see page 66 in Eisenbud and Harris (1999).

Question 5. *What is the multiplicity of the origin $(0,0,0)$ as a zero of the polynomial equations*

$$x^5 + y^3 + z^3 = x^3 + y^5 + z^3 = x^3 + y^3 + z^5 = 0?$$

Solution. If I is the ideal generated by $x^5 + y^3 + z^3$, $x^3 + y^5 + z^3$ and $x^3 + y^3 + z^5$ in $\mathbb{Q}[x, y, z]$, then the multiplicity of the origin is

$$\dim_{\mathbb{Q}} \frac{\mathbb{Q}[x, y, z]_{\langle x, y, z \rangle}}{I \cdot \mathbb{Q}[x, y, z]_{\langle x, y, z \rangle}}.$$

It follows that the multiplicity is the vector space dimension of the ring $\mathbb{Q}[x, y, z]/\varphi^{-1}(I \cdot \mathbb{Q}[x, y, z]_{\langle x, y, z \rangle})$ where $\varphi: \mathbb{Q}[x, y, z] \rightarrow \mathbb{Q}[x, y, z]_{\langle x, y, z \rangle}$ is the natural map. Moreover, we can express this using ideal quotients:

$$\varphi^{-1}(I \cdot \mathbb{Q}[x, y, z]_{\langle x, y, z \rangle}) = (I : (I : \langle x, y, z \rangle^\infty)).$$

Carrying out this calculation in MACAULAY2, we obtain:

```

i1 : S = QQ[x, y, z];
i2 : I = ideal(x^5+y^3+z^3, x^3+y^5+z^3, x^3+y^3+z^5);
o2 : Ideal of S
i3 : multiplicity = degree(I : saturate(I))
o3 = 27

```

Flat Families. Non-reduced schemes arise naturally as the flat limit of a family of reduced schemes. Exercise III-68 in Eisenbud and Harris (1999) illustrates how a family of skew lines in \mathbb{P}^3 gives a double line with an embedded point.

Question 6. *Let L and M be the lines in $\mathbb{P}_{k[t]}^3$ given by $x = y = 0$ and $x - tz = y + t^2w = 0$ respectively. Show that the flat limit as $t \rightarrow 0$ of the union $L \cup M$ is the double line $x^2 = y = 0$ with an embedded point of degree 1 located at the point $(0 : 0 : 0 : 1)$.*

Solution. We find the flat limit by saturating the intersection ideal:

```

i1 : PP3 = QQ[x, y, z, w];
i2 : S = QQ[t, x, y, z, w];
i3 : phi = map(PP3, S, 0 | vars PP3 );
o3 : RingMap PP3 <--- S
i4 : L = ideal(x, y);
o4 : Ideal of S
i5 : M = ideal(x-t*z, y-t^2*w);
o5 : Ideal of S
i6 : X = intersect(L, M);
o6 : Ideal of S
i7 : Xzero = trim phi substitute(saturate(X, t), {t => 0})
          2      2
o7 = ideal (y*z, y , x*y, x )
o7 : Ideal of PP3

```

This is the union of a double line and an embedded point of degree 1.

```

i8 : use PP3;
i9 : intersect(ideal(x^2, y), ideal(x, y^2, z))
          2      2
o9 = ideal (y*z, y , x*y, x )
o9 : Ideal of PP3
i10 : degree( ideal(x^2, y) / ideal(x, y^2, z))
o10 = 1

```

Bézout’s Theorem. Bézout’s theorem (Theorem III-78 in Eisenbud and Harris, 1999) fails without the Cohen-Macaulay hypothesis. Following Exercise III-81 in Eisenbud and Harris (1999), we illustrate this in MACAULAY2.

Question 7. Find irreducible closed subvarieties X and Y in \mathbb{P}^4 with

$$\begin{aligned} \operatorname{codim}(X \cap Y) &= \operatorname{codim}(X) + \operatorname{codim}(Y) \text{ and} \\ \deg(X \cap Y) &> \deg(X) \cdot \deg(Y). \end{aligned}$$

Solution. We show that the assertion holds when X is the cone over the nonsingular rational quartic curve in \mathbb{P}^3 and Y is a two-plane passing through the vertex of the cone. The computation is done as follows:

```

i1 : S = QQ[a, b, c, d, e];
i2 : quarticCone = trim minors(2,
          matrix{{a, b^2, b*d, c}, {b, a*c, c^2, d}})
          3      2      2      2      3      2
o2 = ideal (b*c - a*d, c - b*d , a*c - b d, b - a c)
o2 : Ideal of S
i3 : plane = ideal(a, d);
o3 : Ideal of S
i4 : codim quarticCone + codim plane == codim (quarticCone + plane)
o4 = true

```

```

i5 : (degree quarticCone) * (degree plane)
o5 = 4
i6 : degree (quarticCone + plane)
o6 = 5

```

Constructing Blow-ups. The blow-up of a scheme X along a subscheme Y can be constructed from the Rees algebra associated to the ideal sheaf of Y in X ; see Theorem IV-22 in Eisenbud and Harris (1999). Gröbner basis techniques allow one to express the Rees algebra in terms of generators and relations. We demonstrate this by solving Exercise IV-43 in Eisenbud and Harris (1999).

Question 8. Find the blow-up of the affine plane $\mathbb{A}^2 = \text{Spec}(k[x, y])$ along the subscheme defined by $\langle x^3, xy, y^2 \rangle$.

Solution. We first provide a general function which given an ideal and a list of variables returns the ideal of relations for the Rees algebra.

```

i1 : blowUpIdeal = method();
i2 : blowUpIdeal(Ideal, List) := (I, L) -> (
    r := numgens I;
    S := ring I;
    kk := coefficientRing S;
    n := numgens S;
    y := symbol y;
    St := kk[t, gens S, y_1..y_r, MonomialOrder => Eliminate 1];
    phi := map(St, S, submatrix(vars St, {1..n}));
    F := phi gens I;
    local J;
    J = ideal apply(1..r, j -> y_j - t*F_(0, (j-1)));
    J = ideal selectInSubring(1, gens gb J);
    if (#L < r) then error "not enough variables";
    R := kk[gens S, L];
    theta := map(R, St, 0 | vars R);
    theta J
);

```

Applying the function to our specific case yields:

```

i3 : S = QQ[x, y];
i4 : I = ideal(x^3, x*y, y^2);
o4 : Ideal of S
i5 : blowUpIdeal(I, {A, B, C})
o5 = ideal (y*B - x*C, x*B^2 - A*C, x^2*B - y*A, x^3*C - y^2*A)
o5 : Ideal of QQ [x, y, A, B, C]

```

A Classic Blow-up. We consider the blow-up of the projective plane \mathbb{P}^2 at a point. Many related examples appear in section IV.2.2 of Eisenbud and Harris (1999).

Question 9. Show that the following varieties are isomorphic.

(a) the image of the rational map from \mathbb{P}^2 to \mathbb{P}^4 given by

$$(r : s : t) \mapsto (r^2 : s^2 : rs : rt : st);$$

(b) the blow-up of the plane \mathbb{P}^2 at the point $(0 : 0 : 1)$;

(c) the determinantal variety defined by the 2×2 minors of the matrix

$$\begin{bmatrix} a & c & d \\ b & d & e \end{bmatrix}$$

where $\mathbb{P}^4 = \text{Proj}(k[a, b, c, d, e])$.

This surface is called the cubic scroll in \mathbb{P}^4 .

Solution. We find the ideal in part (a) by elimination theory.

```
i1 : PP4 = QQ[a, b, c, d, e];
i2 : S = QQ[r, s, t, A..E, MonomialOrder => Eliminate 3];
i3 : I = ideal(A - r^2, B - s^2, C - r*s, D - r*t, E - s*t);
o3 : Ideal of S
i4 : phi = map(PP4, S, (matrix{{0, 0, 0}}**PP4) | vars PP4)
o4 = map(PP4,S,{0, 0, 0, a, b, c, d, e})
o4 : RingMap PP4 <--- S
i5 : surfaceA = phi ideal selectInSubring(1, gens gb I)
o5 = ideal (c*d - a*e, b*d - c*e, a*b - c^2)
o5 : Ideal of PP4
```

We determine the surface in part (b) by constructing the blow-up of \mathbb{P}^2 in $\mathbb{P}^2 \times \mathbb{P}^1$ and then projecting its Segre embedding from \mathbb{P}^5 into \mathbb{P}^4 . Notice that its image under the Segre map lies on a hyperplane in \mathbb{P}^5 .

```
i6 : R = QQ[t, x, y, z, u, v, MonomialOrder => Eliminate 1];
i7 : blowUpIdeal = ideal selectInSubring(1, gens gb ideal(u-t*x, v-t*y))
o7 = ideal(y*u - x*v)
o7 : Ideal of R
i8 : PP2xPP1 = QQ[x, y, z, u, v];
i9 : psi = map(PP2xPP1, R, 0 | vars PP2xPP1);
o9 : RingMap PP2xPP1 <--- R
i10 : blowUp = PP2xPP1 / psi(blowUpIdeal);
i11 : PP5 = QQ[A, B, C, D, E, F];
i12 : segre = map(blowUp, PP5, matrix{{x*u, y*u, z*u, x*v, y*v, z*v}});
o12 : RingMap blowUp <--- PP5
i13 : ker segre
o13 = ideal (B - D, C*E - D*F, D^2 - A*E, C*D - A*F)
o13 : Ideal of PP5
i14 : theta = map( PP4, PP5, matrix{{a, c, d, c, b, e}})
o14 = map(PP4,PP5,{a, c, d, c, b, e})
o14 : RingMap PP4 <--- PP5
```

```

i15 : surfaceB = trim theta ker segre
                                     2
o15 = ideal (c*d - a*e, b*d - c*e, a*b - c )
o15 : Ideal of PP4

Finally, we compute the surface in part (c) and apply a permutation
of the variables to obtain the desired isomorphisms

i16 : determinantal = minors(2, matrix{{a, c, d},{b, d, e}})
                                     2
o16 = ideal (- b*c + a*d, - b*d + a*e, - d  + c*e)
o16 : Ideal of PP4
i17 : sigma = map(PP4, PP4, matrix{{d, e, a, c, b}});
o17 : RingMap PP4 <--- PP4
i18 : surfaceC = sigma determinantal
                                     2
o18 = ideal (c*d - a*e, b*d - c*e, a*b - c )
o18 : Ideal of PP4
i19 : surfaceA == surfaceB
o19 = true
i20 : surfaceB == surfaceC
o20 = true

```

Fano Schemes. Our last example concerns the family of Fano schemes associated to a flat family of quadrics. We solve Exercise IV-69 in Eisenbud and Harris (1999).

Question 10. *Consider the one-parameter family of quadrics*

$$Q = V(tx^2 + ty^2 + tz^2 + w^2) \subseteq \mathbb{P}_{k[t]}^3 = \text{Proj}(k[t][x, y, z, w]).$$

As the fiber Q_t tends to the double plane Q_0 , what is the flat limit of the Fano scheme $F_1(Q_t)$ of lines lying on these quadric surfaces?

Solution. We first make the homogeneous coordinate ring of the ambient projective 3-space and the ideal of our family of quadrics.

```

i1 : PP3overBase = QQ[t, x, y, z, w];
i2 : Qt = ideal(t*x^2+t*y^2+t*z^2+w^2);
o2 : Ideal of PP3overBase

```

We construct an indeterminate line in $\mathbb{P}_{\mathbb{Q}[t]}^3$ by adding parameters u, v and two points $(A: B: C: D)$ and $(E: F: G: H)$. The map ϕ sends the variables to the coordinates of the general point on this line.

```

i3 : S = QQ[t, u, v, A..H];
i4 : phi = map(S, PP3overBase, matrix{{t} |
      u*matrix{{A, B, C, D}}+v*matrix{{E, F, G, H}}});
o4 : RingMap S <--- PP3overBase

```

The indeterminate line is contained in our family of quadrics if and only if $\phi(tx^2 + ty^2 + tz^2 + w^2)$ vanishes identically in u and v . Thus, we extract the coefficients of u and v .

```

i5 : imageOfQt = phi Qt;
o5 : Ideal of S
i6 : coeff = (coefficients({1,2}, gens imageOfQt))_1;
      1      3
o6 : Matrix S <--- S

```

We no longer need the variables u and v .

```

i7 : Sprime = QQ[t, A..H];
i8 : coeff = substitute(coeff, Sprime);
      1      3
o8 : Matrix Sprime <--- Sprime
i9 : Sbar = Sprime / (ideal coeff);

```

To move to the Grassmannian over $\mathbb{Q}[t]$, we introduce a polynomial ring in 6 new variables corresponding to the minors of the matrix $\begin{bmatrix} A & B & C & D \\ E & F & G & H \end{bmatrix}$. The map ψ sends the new variables a, \dots, f to the appropriate minor, regarded as elements of Sbar .

```

i10 : PP5overBase = QQ[t, a..f];
i11 : psi = matrix{{Sbar_"t"}} | substitute(
      exteriorPower(2, matrix{{A, B, C, D}, {E, F, G, H}}), Sbar);
      1      7
o11 : Matrix Sbar <--- Sbar
i12 : fanoOfQt = trim ker map(Sbar, PP5overBase, psi);
o12 : Ideal of PP5overBase

```

We next determine the limit as t tends to 0.

```

i13 : fanoOfQ0 = trim substitute(saturate(
      fanoOfQt, t), {t => 0})
      2 2 2
o13 = ideal (e*f, d*f, d*e, a*e + b*f, d , f , e , c*d - b*e + a*f,
      2 2 2
      b*d + c*e, a*d - c*f, a + b + c )
o13 : Ideal of PP5overBase
i14 : degree( ideal(d, e, f, a^2+b^2+c^2) / fanoOfQ0)
o14 = 2

```

We see that $F_1(Q_0)$ is supported on the plane conic $\langle d, e, f, a^2 + b^2 + c^2 \rangle$ and $F_1(Q_0)$ is not reduced — it has multiplicity two.

From section IV.3.2 in Eisenbud and Harris (1999), we know that $F_1(Q_1)$ is the union of two conics lying in complementary planes. We verify this as follows:

```

i15 : fanoOfQ1 = trim substitute(saturate(
      fanoOfQt, t), {t => 1})
      2 2 2
o15 = ideal (a*e + b*f, d + e + f , c*d - b*e + a*f, b*d + c*e,
      2 2 2
      a*d - c*f, c + e + f , b*c + d*e, a*c - d*f, b - e ,
      2 2
      a*b + e*f, a - f )

```

```

o15 : Ideal of PP5overBase
i16 : fanoOfQ1 == intersect(ideal(c-d, b+e, a-f, d^2+e^2+f^2),
                           ideal(c+d, b-e, a+f, d^2+e^2+f^2))
o16 = true

```

Thus, $F_1(Q_0)$ is the double conic obtained when the two conics in $F_1(Q_1)$ move together.

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